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Strictly local growth of Penrose patterns*

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Abstract. Non-locality in the sense of Penrose does not imply that no local growth algorithms can exist which generate global tilings in an aperiodic species. The unsuccessful search (until now) for the species of Penrose tilings (S_P) provided the motivation to develop growth algorithms which accept violation of matching rules. These are described and statistical data are given.

1. Introduction

In 1988 Onoda *et al* (cf [5]) published an algorithm for the growth of perfect Penrose tilings (PPT) by first building an imperium of a given patch (cluster) using local decisions and then adding a fat tile at a special edge of the imperium. In a reply Jarić and Ronchetti in [7] have pointed out that this should not be called a local growth algorithm, as the decision of whether a given patch is an imperium or not is surely not local, because one has to investigate the whole surface of the patch, an opinion which is supported by Olami [10, 11]. We call the suggested algorithm a *global–local algorithm*, because the decision where the next tile is added is a global one, whereas it is locally decided which tile is added. It is possible to approximate the algorithm by Onoda *et al* using probabilistic decisions. If an edge is forced, the probability of adding a tile is unity. If not, the probability is zero unless it is a special type of edge, for which the probability is as small as desired. Such an algorithm is local–local, producing defect Penrose tilings. For more details see [7, 12]. Also in the latter paper the importance of special defective initial patches which have an imperium covering the whole plane is shown, so the non-local part of the algorithm by Onoda *et al* is not necessary. We have to contradict the common opinion, which Penrose has actually proved in [6], that a *local–local algorithm* for PPT is impossible. After a long and unsuccessful search for such an algorithm we too believe that it is impossible, but the property of Penrose non-locality defined according to [6] does not exclude such an algorithm. In a second part we present several algorithms for the growth of defect Penrose tilings (DPT), in which we have tried to make the first term local in our sense of a global–local algorithm to truly local–local algorithms. A quite different but rather general approach to the problem can be found in the paper by Moody and Patera [14].

2. Non-locality

A *species* $S = S(\mathcal{F}, M)$ is the set of all global face-to-face tilings whose tiles belong to the family $\mathcal{F}$ of prototiles under the building restriction or method M. A species S is called *aperiodic*, if and only if all tilings in S are non-periodic.

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According to the classification of Penrose we shall call a patch *correct* if it appears in a tiling of the actual species, otherwise it is said to be *not correct.* Trying to generalize the so-called *non-local* property, which Penrose showed to hold for the species S_P of all perfect Penrose tilings (PPTs) [6], one might suggest the following definition.

Definition 1 (NLP′). For every $\rho > 0$ there exists a correct patch $\mathcal{A}$ and two tiles T_1 and T_2 with $\mathrm{dist}(T_1, T_2) > \rho$ such that $\mathcal{A} \cup \{T_1\}$ and $\mathcal{A} \cup \{T_2\}$ are both correct patches but $\mathcal{A} \cup \{T_1, T_2\}$ is not correct.

But the species given by the crystallographic tiling in figure 1 would be non-local by this definition, as you obtain a correct patch adding the tiles T_1 or T_2 in figure 2, but if the (arbitrarily large) number of tiles in between is odd the patch containing T_1 and T_2 is not correct.

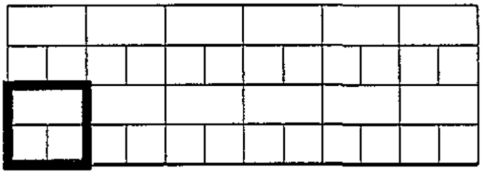

Figure 1. A crystallographic tiling.

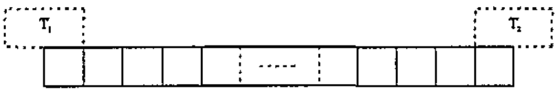

Figure 2. Should this cause non-locality?

Hence we suggest another definition (which still holds for S_P as shown by Penrose). For this purpose we define $A(\rho)$ to be the set of all correct patches, covering a disc of radius ρ. The new version of non-locality then reads as follows:

Definition 2 (NLP). For every $\rho > 0$ there is a patch $\mathcal{A} \in A(\rho)$ and two tiles T_1 and T_2 with $\mathrm{dist}(T_1, T_2) > \rho$ such that $\mathcal{A} \cup \{T_1\}$ and $\mathcal{A} \cup \{T_2\}$ are correct patches, but $\mathcal{A} \cup \{T_1, T_2\}$ is not correct.

Species fulfilling this definition will be called *Penrose-non-local.* We want to point out that a species of locally isometric tilings, which contains one crystallographic tiling and therefore only this tiling, is not Penrose-non-local. We would like to establish another definition of non-locality, concentrating on species built with the help of local matching rules (LMR). (It has also been proved by Penrose, but only implicitly, that S_P has this property.) Therefore we say a patch is called *legal,* if it satisfies local matching rules. Hence a patch can be 'legal and correct', 'legal and not correct', or 'not legal and (therefore) not correct'.

Definition 3 (NL). A species $S = S(\mathcal{F}, \mathrm{LMR})$ is called *non-local* (NL), if and only if for every $\rho > 0$ there exist a patch $\mathcal{A} \in A(\rho)$, a tile T, and a patch $\mathcal{B}$, such that the following holds:

$$\mathcal{A} \cup (T + \rho\mathbb{B}_2) \subset \mathcal{B} \text{ and } \mathcal{B} \text{ is a legal patch, but } (\mathcal{A} \cup \{T\}) \notin A(\rho)$$

where $\mathbb{B}_2$ is the unit disc.

Obviously, a Penrose-non-local species generated with local matching rules is non-local in this sense.

But non-locality does not imply that it is impossible to find any local–local algorithm for tilings of such species defined by local matching rules. Non-locality only excludes that the local matching rules automatically avoid wrong choices. It might be possible to develop a growth algorithm including a catalogue of local rules, however complex this might be, defining the choices to be made under certain local conditions.

One example is the following. In the tiling of figure 1 you can shift each layer of rectangles by one square without changing the rest of the tiling. Now randomly choose for every layer whether to shift or not. This yields a species with infinitely many different local isometry classes. It is obviously non-local according to each of the previous definitions, but one could easily find local–local algorithms which grow perfect tilings of this species.

Another (not so monster-like and three-dimensional) example is the aperiodic species of all monohedral tilings consisting of the 'SCD tile'. The SCD tile was successively developed by Schmitt and Conway, and this paper's last author [13]. Although no SCD tiling permits any translation at all, they are built up of 'shiftable' layers and arguments for non-locality and local–local algorithms work analogously. It should be noticed that there are homogeneous SCD species, e.g. species consisting of only one local isometry class.

3. Growth algorithms for defect Penrose tilings

Any growth algorithm can be described according to the following two basic questions: '*where?*' (which part of an actual patch does grow?) and '*what?*' (how will the patch be continued?). For example the algorithm of Onoda *et al* [5] gives an answer to *where*: (i) choose randomly a forced edge, (ii) take a special edge at a corner of a patch with an unforced surface (surely a global answer). And to *what*: (i) add the forced tile, (ii) add a fat rhomb legally (a local answer). Therefore we call this a *global–local algorithm.*

As mentioned above, we strongly believe that a local–local algorithm is impossible for the species S_P. So one might say that we started at the other end of the scale by trying to develop rather simple growth algorithms which generate defect Penrose tilings (DPT) which already have a relatively low frequency of defects, algorithms which do not run into dead ends. We were interested in which set of defects was necessary to enable such algorithms to work. Onoda *et al* in [5] and Jarić and Ronchetti in [7] agreed that a study of defects occurring (naturally?) would be useful. So our algorithms only accept defects out of a small set of specially selected kinds of defects in Penrose tilings. We do not allow a mixing up of the two classes of vertices into which the set of vertices of a Penrose tiling can be subdivided (see, for example, [2]), but we accept edges where two oppositely oriented edges of tiles (having the same vertex class at each end) meet. With these so-called *green* edges we build twelve defective vertex stars and only these phason defects were allowed in our growth algorithms, and only when none of the eight different vertex stars occurring in PPTs fitted (see figure 3). (As mathematicians we prefer the geometrically simplest tiles and so all our growth algorithms work on Penrose tilings consisting of Robinson triangles with edge lengths $1, \tau^{-1}, \tau$ ($\tau := \frac{1}{2} + \frac{1}{2}\sqrt{5}$).)

In contrast to Onoda *et al* our growth algorithms do not select an edge on the surface and add a tile, but select a vertex on the surface and complete the vertex star. As to the *what* question, we try to minimize the number of green edges at the vertex itself and the two neighbouring vertices on the surface using a list of priorities for vertex stars for

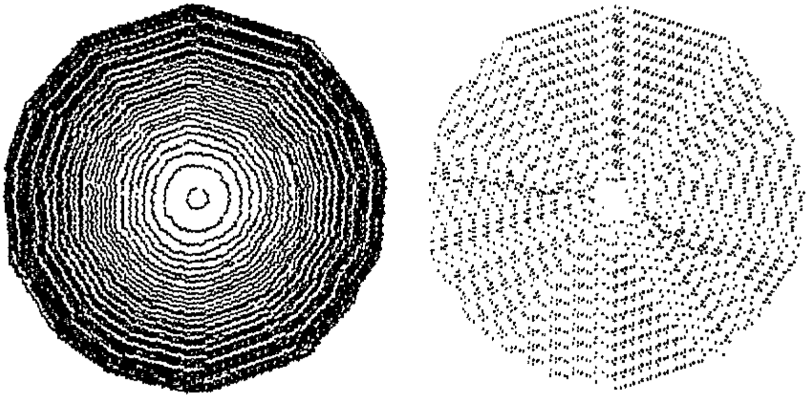

Figure 3. Development of the surface and distribution of green edges in a patch grown with the 'oldest' algorithm.

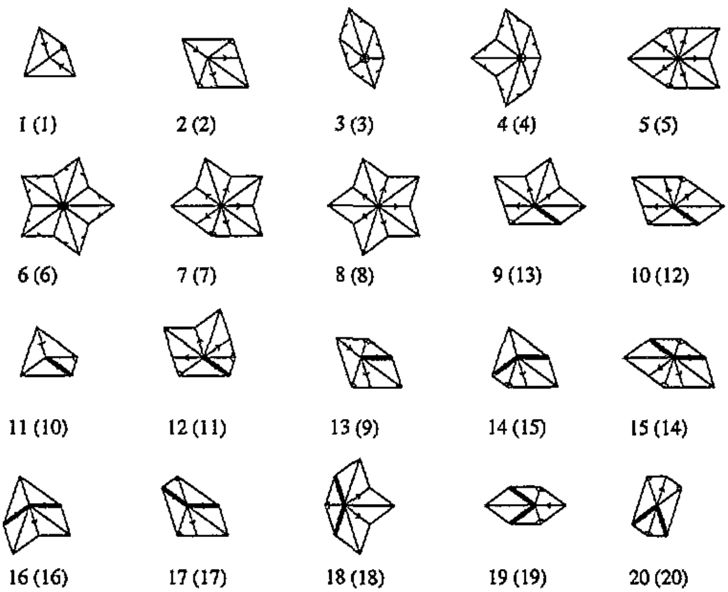

Figure 4. The permitted vertex stars.

all fitting vertex stars (legal and defective), which force the minimal number of green edges. This is not changed for any of the listed algorithms. For the list of priorities, see the numbers in brackets in figure 4. Of all fitting vertex stars with minimal number of forced green edges we choose the one with the lowest number. (If this did not force the position of each tile from the completed vertex star, it was chosen randomly). For the legal vertex stars the priority was chosen according to the volume of their acceptance domain in the orthogonal space. The method inspired by Baake and Joseph in [9] to define via representation in orthogonal space an elastic energy measure for defective vertex stars unfortunately does not work in our context, because only two defective vertex stars would cause energetic costs. The theoretical 'acceptance domains' of most defective vertex stars are points or lines and therefore would have measure 0, unlike the defective vertex stars

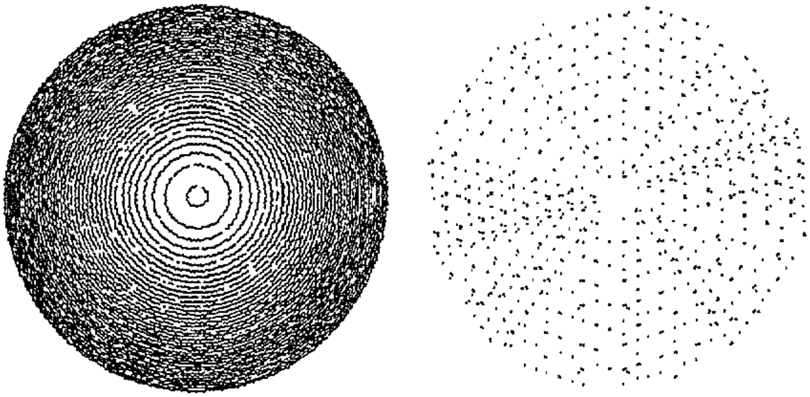

Figure 5. Development of the surface and distribution of green edges in a patch grown with the 'nearest' algorithm.

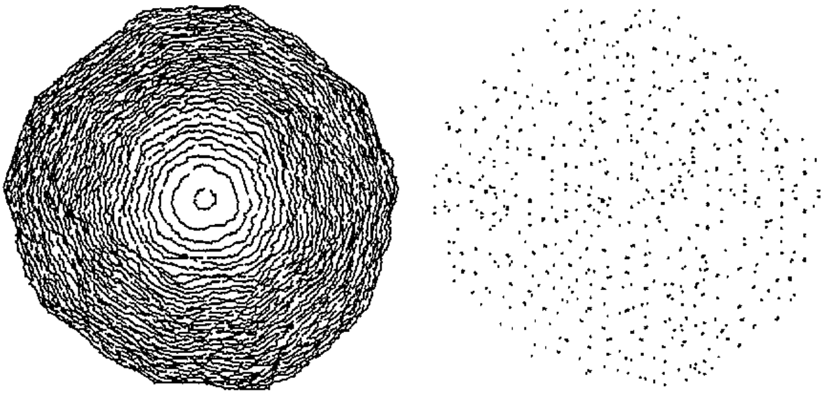

Figure 6. Development of the surface and distribution of green edges in a patch grown with the 'line' algorithm.

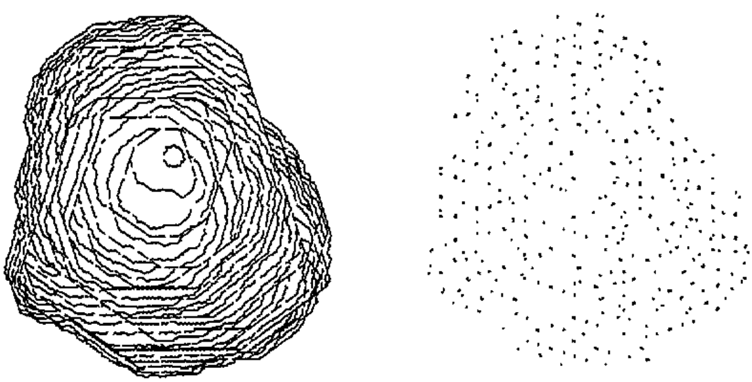

Figure 7. Development of the surface and distribution of green edges in a patch grown with the 'concave' algorithm.

in the octagonal tilings considered in the work mentioned above. As a matter of fact a change in the priority list did not have any great effects on the global structure of the

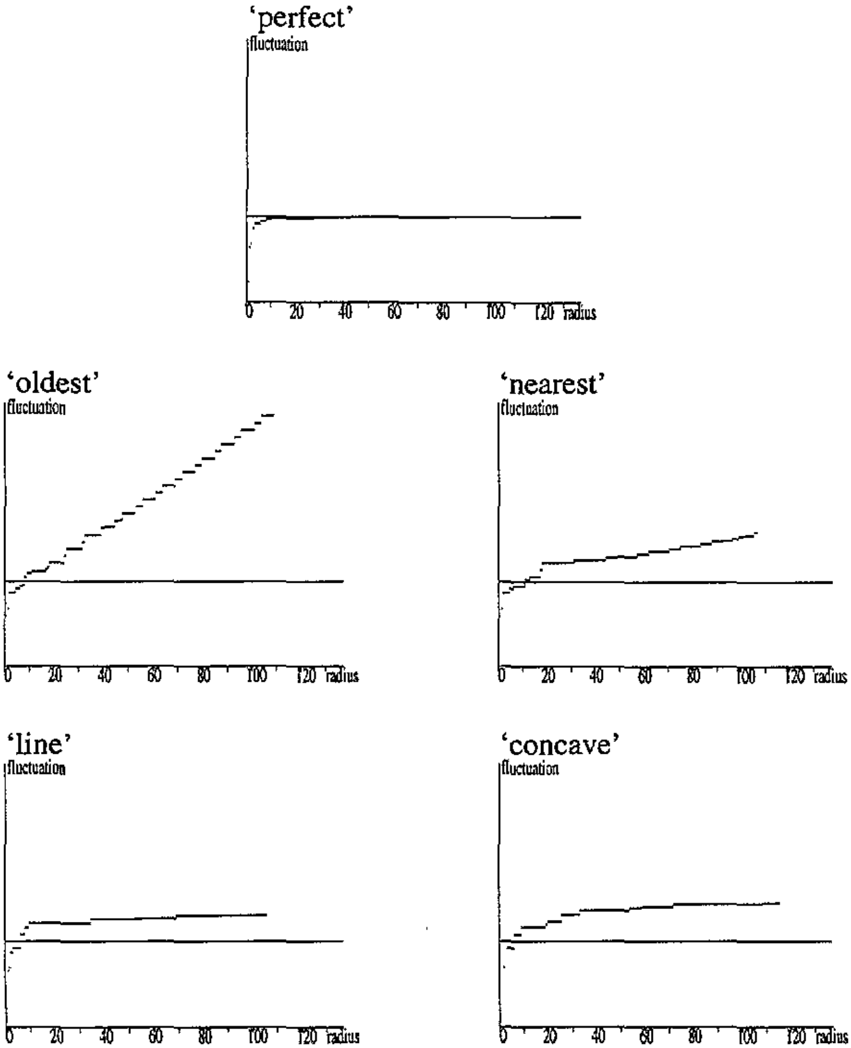

Figure 8. The fluctuation in orthogonal space with respect to the radius for a perfect Penrose tiling and for patches grown by the different algorithms.

grown patch or on the number of defects in it. Again we would like to point out that the aim of this paper is to find out how many defects and how many types of defects have to be accepted if simple local–local algorithms are to lead to tilings of the whole plane.

An essential question concerning growth algorithms is if shifts of the strip are bounded in the orthogonal space, which would reveal that the patterns are quasicrystallographic rather than crystallographic or amorphous. We hope that for two of our algorithms these fluctuations are bounded, but unfortunately we are not able to prove this. (As a consequence the broadening of peaks in the Fourier spectrum would be bounded, too, as is stated in [3, 10]. Actually, neither could not prove that their algorithms exclude unlimited shifts of the strip either, although they seem to be very sure about this.) In figure 8 the growth of the acceptance domain in the orthogonal space is shown, as it depends on the diameter of the patches. Only in the case of the 'oldest' algorithm does it seem to be quite obvious that shifts are unlimited.

The *where* question becomes important when growing large patches, because uncontrolled growth can lead to holes. The existence of choices (or the property of non-locality) means, almost inescapably, that two parts of a patch, which have grown into different directions, do not fit if they grow together again. So it is reasonable to try to keep the patch 'convex'. The first, and not new, idea to achieve this has been inspired by the physical idea that the time between the moment when an atom has settled on the surface and the moment when it is surrounded by others is approximately the same for all atoms. It is a matter of taste to call this algorithm global or local.

Algorithm 1 ('oldest'). First make a list of all vertices on the surface.

Choose the vertex at the top of the list and complete it with the minimal number of green edges and according to the priority list. Add the newly generated vertices at the bottom of the list and remove all vertex stars from it which have now been completed.

One feature of this algorithm is that the outer shape of the grown patches mostly looks like a decagon with edges perpendicular to the orientation of the tiles' edges. This decagonal shape was also built by an algorithm for atomistic growth of decagonal quasicrystals of Szeto and Wang [8], which is not too surprising, since our 'oldest' algorithm can be considered as a further development of their 'layer-by-layer' idea, using a minimum of memory. The patches built by their algorithm (or a previous one by Minchau *et al* in [4]) or ours look rather like a multi-twining crystal, so we expect the fluctuation of the strip in orthogonal space not to be bounded.

The canonical geometric solution to the convexity problem is realized in the second algorithm:

Algorithm 2 ('nearest'). Choose the vertex on the surface which is nearest to the origin, and complete it with the minimal number of green edges and according to the priority list.

This surely is a global–local algorithm, because it needs an origin (globally defined). On the other hand it is more local than the algorithm of Onoda *et al*, as for one vertex the distance from the origin does not change, whereas the property of being unforced does change while growing.

Trying to make this 'nearest' algorithm more local, we came to the idea of approximating a part of the circle deciding which vertex is the nearest by the straight line approximating a part of a fixed size of the surface. Choosing this part of the surface randomly leads to the problems discussed above. To avoid this problem we choose the part walking around the surface with a fixed angle of $2\pi\tau^{-1}$. This guarantees a strongly balanced distribution of the parts on the surface, while keeping the local situation near to that of a real random choice. Given such a local part of the surface, we detected the vertices 'nearest to the origin' by their position relatively to the straight line approximating this part of the surface.

Algorithm 3 ('line'). Define an orientation of the surface of the given patch and choose one starting direction from the origin.

Let E_0 be the vertex on the surface which is next to the beam into the chosen direction. Call the next six free vertex stars in positive direction $E_1, \ldots, E_6$. Find the line g for which the sum of its squared distances to $E_0, \ldots, E_6$ is minimal. To find the 'inner side' of g, define s to be the centre of gravity of the second corona of E_3 and the vertices on the same side of g as s to be inner vertices. Now gradually complete the inner vertices out of

$E_1, \ldots, E_5$ with the maximal distance to g as long as the distance is greater than 0.2. This completion is done according to the same rules as in the other algorithm. Afterwards rotate the direction by $2\pi\tau^{-1}$ (and iterate the procedure).

As it is possible for small starting patches for this algorithm to stop (because, according to it, no vertex E_0 causes a completion), we decrease the value of 0.2 for every turn of the algorithm without a completion, and set it back to 0.2 as soon as one vertex was completed. But this is only necessary for small patches. Starting with patches of a radius greater than 16, it is not necessary to use this additional rule.

This algorithm is almost a local–local algorithm as it uses an origin (global), but only in order to get a very strongly balanced distribution on the surface of the patch.

The following algorithm is local–local in the strongest sense.

Algorithm 4 ('concave'). First choose an orientation of the surface and arbitrarily a free vertex star, called E_0.

Let E_1, E_2 and E_3 be the three successive neighbours of E_0 in positive orientation. Complete E_0 according to the same rules as in the other algorithms, if and only if the surface is strictly concave at this vertex. Now go back from E_3 on the—eventually new—surface to the next but one vertex and call it E_0 (and iterate the procedure).

If there was no vertex star completed within 100 steps, change the condition 'strictly concave' into 'concave', until one vertex star is completed.

The least local part of this algorithm is the decision of whether there was no completion during the last 100 steps. Hence this is a local–local algorithm.

4. Conclusions

Growing defective Penrose patterns without controlling the shape of the growing patch cannot work. However, we have shown that it is possible to steer the growth by strictly local rules. The frequency of defects in our grown tilings is relatively high, because we always try to use the simplest algorithm of one kind to show its characteristics. The frequency can be decreased easily by taking some more complex rules for the choices of the vertex star, which is chosen from the permitted ones (e.g. make a list of possible local surfaces and say what to add in every case). It is also possible to prefer forced vertices in a local part of the surface. Enlarging this part step by step yields a continuous transition to an algorithm like the one of Onoda *et al* with defect-frequency zero.

Most of the once-built defects appearing in the tilings can be 'moved' by flips, i.e. by reflecting a hexagon of tiles. Hence it is possible to discard a high percentage of all defects by cancelling two defects in the same Conway 'worm', or moving them through the surface out of the patch. So we assume that if a rule for local moving of defects is built into the algorithms, the frequency of defects could be essentially decreased.

We have worked hard in preparing a transfer of the above algorithms to the three-dimensional case, especially for application on the species of $\{A, B, C, K\}$-tilings. Since our grant has been cut down drastically we cannot pursue this task any longer. But we still think it would make sense.

Acknowledgments

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Appendix.

Table A1. Basic data of the grown patches.

Algorithm	Oldest	Nearest	Line	Concave
Number of grown patches	70	70	70	70
Max. number of vertex completions	38 000	40 000	40 000	30 000
Percentage of defect edges	1.804%	0.612%	0.540%	0.468%
Percentage of defect vertex stars	10.276%	3.591%	3.212%	2.821%
Number of types of defect vertex stars	10	6	8	8
Sums over all grown patches:				
Vertex completions	222 000	234 000	228 000	212 005
Choices	22 963	12 301	8 915	6 305
Inner edges	766 906	771 666	738 662	700 956
Defect edges	13 833	4 722	3 990	3 278
Inner vertices	249 431	252 279	242 070	229 857
Defect vertex stars	25 632	9 060	7 776	6 485
Tiles	517 545	519 457	496 662	471 169

Table A2. Frequency of the vertex stars in the grown patches and in a perfect Penrose tiling.

Vertex star	Oldest	Nearest	Line	Concave	Perfect†
1	33.386%	36.569%	37.011%	37.252%	38.197%
2	18.459%	22.024%	22.178%	22.271%	23.607%
3	13.793%	14.117%	14.286%	14.546%	14.590%
4	10.423%	9.854%	9.480%	9.135%	9.017%
5	4.522%	4.422%	4.828%	5.185%	5.573%
6	3.528%	3.738%	3.752%	3.780%	4.033%
7	5.584%	5.180%	4.431%	3.981%	3.444%
8	0.028%	0.506%	0.823%	1.028%	1.540%
9	0.529%	0.002%	0.031%	0.017%	
10	0.001%	0.002%	0.053%	0.034%	
11	4.101%	1.392%	1.327%	1.276%	
12	0.111%	0.394%	0.244%	0.107%	
13	4.948%	1.762%	1.537%	1.384%	
14	0.0004%	0.000%	0.000%	0.001%	
15	0.000%	0.039%	0.019%	0.002%	
16	0.000%	0.000%	0.0004%	0.000%	
17	0.575%	0.000%	0.001%	0.000%	
18	0.002%	0.000%	0.000%	0.001%	
19	0.005%	0.000%	0.000%	0.000%	
20	0.004%	0.000%	0.000%	0.000%	

† According to [1].

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